

On SA, CA, and GA numbers

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Abstract Gronwall's function G is defined for $n > 1$ by $G(n) = \frac{\sigma(n)}{n \log \log n}$ where $\sigma(n)$ is the sum of the divisors of n . We call an integer $N > 1$ a *GA1 number* if N is composite and $G(N) \geq G(N/p)$ for all prime factors p of N . We say that N is a *GA2 number* if $G(N) \geq G(aN)$ for all multiples aN of N . In [3], we used Robin's and Gronwall's theorems on G to prove that the Riemann Hypothesis (RH) is true if and only if 4 is the only number that is both GA1 and GA2. In the present paper, we study GA1 numbers and GA2 numbers separately. We compare them with superabundant (SA) and colossally abundant (CA) numbers (first studied by Ramanujan). We give algorithms for computing GA1 numbers; the smallest one with more than two prime factors is 183783600, while the smallest odd one is 1058462574572984015114271643676625. We find nineteen GA2 numbers ≤ 5040 , and prove that a GA2 number $N > 5040$ exists if and only if RH is false, in which case N is even and $> 10^{8576}$.

Keywords Algorithm · colossally abundant · Gronwall's theorem · prime factor · Riemann Hypothesis · Robin's inequality · sum-of-divisors function · superabundant

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1 Introduction

The *sum-of-divisors function* σ is defined by

$$\sigma(n) := \sum_{d|n} d.$$

For example, $\sigma(4) = 7$.

In 1913, Gronwall [7] found the maximal order of σ .

Theorem 1 (Gronwall) *The function*

$$G(n) := \frac{\sigma(n)}{n \log \log n} \quad (n > 1)$$

satisfies

$$\limsup_{n \rightarrow \infty} G(n) = e^\gamma = 1.78107 \dots,$$

where γ is the Euler-Mascheroni constant.

In 1915, Ramanujan proved an asymptotic inequality for Gronwall's function G , assuming the Riemann Hypothesis (RH). Ramanujan's result was shown in the second part of his thesis. The first part was published in 1915 [12] while the second part was not published until much later [13].

Theorem 2 (Ramanujan) *If the Riemann Hypothesis is true, then*

$$G(n) < e^\gamma \quad (n \gg 1).$$

Here, $n \gg 1$ means for all sufficiently large n .

In 1984, Robin [14] proved that a stronger statement about the function G is equivalent to RH.

Theorem 3 (Robin) *The Riemann Hypothesis is true if and only if*

$$G(n) < e^\gamma \quad (n > 5040). \tag{1}$$

The condition (1) is called *Robin's inequality*. Table 1 gives the 26 known numbers r for which the reverse inequality $G(r) \geq e^\gamma$ holds (see [17, Sequence A067698]), together with the value of $G(r)$ (truncated). (The “ $a(r)$ ” column is explained in §4, and the “ $Q(r)$ ” column in §7.1.)

In [14] Robin also proved, unconditionally, that

$$G(n) \leq e^\gamma + \frac{0.6482 \dots}{(\log \log n)^2} \quad (n > 1) \tag{2}$$

with equality for $n = 12$. This refines the inequality $\limsup_{n \rightarrow \infty} G(n) \leq e^\gamma$ from Gronwall's theorem.

Recently, the authors [3] used Robin's results to derive another reformulation of RH. Before recalling its statement, we give three definitions and an example.

r	SA	CA	GA1	GA2	Factorization	$\sigma(r)/r$	$G(r)$	$a(r)$	$Q(r)$
3				✓	3	1.333	14.177	0	
4	✓		✓	✓	2^2	1.750	5.357	0	-0.763
5				✓	5	1.200	2.521	0	
6	✓	✓		✓	$2 \cdot 3$	2.000	3.429	0	4.134
8				✓	2^3	1.875	2.561	0	2.091
9					3^2	1.444	1.834	4	7.726
10				✓	$2 \cdot 5$	1.800	2.158	0	1.168
12	✓	✓		✓	$2^2 \cdot 3$	2.333	2.563	0	2.090
16					2^4	1.937	1.899	3	1.348
18				✓	$2 \cdot 3^2$	2.166	2.041	0	1.679
20					$2^2 \cdot 5$	2.100	1.913	3	2.799
24	✓			✓	$2^3 \cdot 3$	2.500	2.162	0	1.185
30					$2 \cdot 3 \cdot 5$	2.400	1.960	2	1.749
36	✓			✓	$2^2 \cdot 3^2$	2.527	1.980	0	1.294
48	✓			✓	$2^4 \cdot 3$	2.583	1.908	0	1.132
60	✓	✓		✓	$2^2 \cdot 3 \cdot 5$	2.800	1.986	0	1.290
72				✓	$2^3 \cdot 3^2$	2.708	1.863	0	1.160
84					$2^2 \cdot 3 \cdot 7$	2.666	1.791	10	1.430
120	✓	✓		✓	$2^3 \cdot 3 \cdot 5$	3.000	1.915	0	1.128
180	✓			✓	$2^2 \cdot 3^2 \cdot 5$	3.033	1.841	0	1.078
240	✓			✓	$2^4 \cdot 3 \cdot 5$	3.100	1.822	0	1.051
360	✓	✓		✓	$2^3 \cdot 3^2 \cdot 5$	3.250	1.833	0	1.044
720	✓				$2^4 \cdot 3^2 \cdot 5$	3.358	1.782	7	1.028
840	✓				$2^3 \cdot 3 \cdot 5 \cdot 7$	3.428	1.797	3	1.065
2520	✓	✓		✓	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	3.714	1.804	0	1.015
5040	✓	✓		✓	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	3.838	1.790	0	1.007

Table 1 The set $\mathcal{R} = \{r \leq 5040 : G(r) \geq e^\gamma = 1.781\dots\}$, which contains the subset $\mathcal{R}_2 = \{N \leq 5040 : N \text{ is GA2}\}$.

A positive integer N is a *GA1 number* if N is composite and the inequality

$$G(N) \geq G(N/p)$$

holds for all prime factors p of N . The first few GA1 numbers are

$$N = 4, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 106, 118, 122, 134, 142, \dots$$

(see [17, Sequence A197638]), and (see §5.3) *the smallest odd GA1 number is*

$$N = 1058462574572984015114271643676625.$$

An integer $N > 1$ is a *GA2 number* if

$$G(N) \geq G(aN)$$

for all multiples aN of N . The nineteen known GA2 numbers (see Theorem 5 and [17, Sequence A197369]) are

$$N = 3, 4, 5, 6, 8, 10, 12, 18, 24, 36, 48, 60, 72, 120, 180, 240, 360, 2520, 5040.$$

Every GA2 number > 5 is even. (*Proof.* If N is odd, then $\sigma(2N) = 3\sigma(N)$, and if N is also GA2, we get

$$\frac{3}{2} \leq \frac{3G(N)}{2G(2N)} = \frac{\log \log 2N}{\log \log N}$$

which implies $N < 7$.)

Finally, a composite number is *extraordinary* if it is both GA1 and GA2.

For example, the smallest extraordinary number is 4. To see this, we first compute $G(4) = 5.357\dots$. Then, as $G(2) < 0$, it follows that 4 is a GA1 number. Since Robin's unconditional bound (2) implies

$$G(n) < e^\gamma + \frac{0.6483}{(\log \log 5)^2} = 4.643\dots < G(4) \quad (n \geq 5),$$

we get that 4 is also GA2. Thus 4 is an extraordinary number.

We can now recall our results from [3, Theorem 6 and Corollary 8].

Theorem 4 (Caveney-Nicolas-Sondow) (i). *The Riemann Hypothesis is true if and only if 4 is the only extraordinary number.*

(ii). *If a counterexample to Robin's inequality exists, then $\mu := \max\{G(n) : n > 5040\}$ also exists and the least number $N > 5040$ with $G(N) = \mu$ is extraordinary.*

If there is an extraordinary number $N > 4$, then N is even (as 5 is not GA1, and no GA2 number > 5 is odd) and $N > 10^{8576}$ (since no GA1 number lies in the interval $[5, 5040]$, and no GA2 number lies in $[5041, 10^{8576}]$ —see Corollary 1).

In the present paper, we study GA1 numbers and GA2 numbers separately.

Preliminary facts about GA1 numbers and GA2 numbers were given in [3]. We recall two of them and make a definition.

Fact 1 (proved by elementary methods in [3, §5]). *The GA1 numbers with exactly two (not necessarily distinct) prime factors are precisely 4 and $2p$, for primes $p \geq 7$.*

We call such GA1 numbers *improper*, while GA1 numbers with at least three (not necessarily distinct) prime factors will be called *proper*.

The smallest proper GA1 number is $v := 183783600$ (see §5.3 and [17, Sequence A201557]). The number v was mentioned in [3, equation (3)] as an example of a (proper) GA1 number that is not a GA2 number (because $G(v) < G(19v)$).

Fact 2 (see [3, Lemma 10]). *If n_0 is a positive integer, then*

$$\limsup_{a \rightarrow \infty} G(an_0) = e^\gamma,$$

which yields the implication

$$N \text{ is GA2} \implies G(N) \geq e^\gamma. \quad (3)$$

An application is an alternate proof that any GA2 number $N > 5$ is even. Namely, as 7 and 9 are not GA2, and as Theorem 2 in [4] says that an integer $n > 9$ is even if $G(n) \geq e^\gamma$, the result follows from (3).

By the method of [3, §5], one can prove two additional properties of GA1 numbers.

Fact 3. *The only prime power GA1 number $N = p^k$ is $N = 4$.*

Fact 4. *A product of three distinct primes $p_1 p_2 p_3$ cannot be a GA1 number. (See §6.2 for a more general result proved by other methods.)*

The rest of the paper is organized as follows. The next subsection establishes notation. In §2 we recall the definitions of superabundant (SA) and colossally abundant (CA) numbers and review some of their properties. In §3 we prove six lemmas needed later. In §4 we give an analog of Theorem 4 for GA2 numbers; in particular, *if RH is false, then infinitely many GA2 numbers exist, and any number $N > 5040$ for which $G(N) = \max\{G(n) : n > 5040\}$ is both GA2 and CA*. In the final four sections we study proper GA1 numbers: §5 compares them with SA and CA numbers, §6 is concerned with their prime factors, §7 gives algorithms for computing them, and §8 estimates the number of them up to x .

1.1 Notation

We let p always denote a prime.

Let $v_p(n)$ denote *the exponent on p in the prime factorization*

$$n = \prod_p p^{v_p(n)}.$$

For $n \geq 1$, we denote *the number of prime factors of n counted with multiplicity* by

$$\Omega(n) := \sum_p v_p(n).$$

For $n > 1$, we denote *the largest prime factor of n* by

$$P(n) := \max\{p : p \mid n\} = \max\{p : v_p(n) > 0\}.$$

As usual, Chebychev's function θ is defined as

$$\theta(x) := \sum_{p \leq x} \log p.$$

2 Review of properties of SA and CA numbers

Superabundant and colossally abundant numbers were first introduced by Ramanujan, who called them generalized highly composite and generalized super highly composite numbers, respectively (cf. [13, §59]). They were rediscovered later by Alaoglu and Erdős [1].

A *superabundant* (SA) number is a positive integer N such that

$$\frac{\sigma(N)}{N} > \frac{\sigma(n)}{n} \quad (0 < n < N).$$

The first few SA numbers are (see [17, Sequence A004394])

$$N = 1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, \dots$$

A *colossally abundant* (CA) number is a positive integer N for which there exists an exponent $\varepsilon > 0$ such that

$$\frac{\sigma(N)}{N^{1+\varepsilon}} \geq \frac{\sigma(n)}{n^{1+\varepsilon}} \quad (n > 1). \quad (4)$$

Such an exponent ε is called a *parameter* of N . The sequence of CA numbers (compare [17, Sequence A004490]) begins

$$N = 1, 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, \dots$$

From (4), it is easy to show that *every CA number is also SA*.

Now let N denote an SA or CA number. Then (see [1, Theorems 1 and 3] or [13, §59])

$$N = 2^{k_2} \cdot 3^{k_3} \cdot 5^{k_5} \cdots p^{k_p} \implies k_2 \geq k_3 \geq k_5 \geq \cdots \geq k_p \quad (5)$$

with $k_p = 1$ unless $N = 4$ or 36 , and [1, Theorem 7]

$$p = P(N) \sim \log N \quad (N \rightarrow \infty). \quad (6)$$

We recall some properties of CA numbers (see [1–3, 6, 9, 13–15]).

Note first that for any fixed positive integer k , the quantity

$$F(t, k) := \frac{\log \left(1 + \frac{1}{t + t^2 + \cdots + t^k} \right)}{\log t}$$

is decreasing on the interval $1 < t < \infty$, and the function $t \mapsto F(t, k)$ maps the interval onto the positive real numbers. Hence, given $\varepsilon > 0$, we may define $x_k = x_k(\varepsilon) > 1$ by

$$F(x_k, k) = \varepsilon. \quad (7)$$

(See [14, p. 189] and [13, §61 and §69].) In particular, if $k = 1$ we set $x = x_1 = x_1(\varepsilon)$, so that

$$F(x, 1) = F(x_1, 1) = \frac{\log \left(1 + \frac{1}{x} \right)}{\log x} = \varepsilon. \quad (8)$$

It is convenient to set $x_0 = +\infty$. From the decreasingness of $F(t, k)$ with respect to both t and k , it follows that the sequence $(x_k)_{k \geq 0}$ is decreasing.

If N is a CA number of parameter ε and p divides N with $v_p(N) = k$, then applying (4) with $n = Np$ yields

$$\varepsilon \geq F(p, k+1) \quad \text{i.e.} \quad p \geq x_{k+1}$$

while, if $k > 0$, applying (4) with $n = N/p$ yields

$$\varepsilon \leq F(p, k) \quad \text{i.e.} \quad p \leq x_k. \quad (9)$$

Let K be the largest integer such that $x_K \geq 2$. Then from (9), for all p 's we have $2 \leq p \leq x_k$ and

$$k = v_p(N) \leq K.$$

Now define the set

$$\mathcal{E} := \{F(p, k) : p \text{ is prime and } k \geq 1\}.$$

Its largest element is

$$\max \mathcal{E} = F(2, 1) = \frac{\log(3/2)}{\log 2} = 0.5849 \dots,$$

and its infimum is

$$\inf \mathcal{E} = \lim_{k \rightarrow \infty} F(p, k) = 0$$

for any fixed prime p .

If $\varepsilon \notin \mathcal{E}$, then no x_k is a prime number and there exists a unique CA number $N = N(\varepsilon)$ of parameter ε ; moreover, N is given by either of the equivalent formulas

$$N = \prod_{p < x} p^{k_p} \quad \text{with } x_{k_p+1} < p < x_{k_p}$$

or

$$N = \prod_{k=1}^K \prod_{p < x_k} p. \quad (10)$$

In particular, if $\varepsilon > \max \mathcal{E}$, then $x = x_1 < 2$, $K = 0$ and $N(\varepsilon) = 1$.

If $\varepsilon \in \mathcal{E}$, then some x_k is prime, and it is highly probable that only one x_k is prime. But (see [6, Proposition 4]), from the theorem of six exponentials it is only possible to show that at most two x_k 's are prime. (Compare [9, p. 538].) Therefore there are either two or four CA numbers of parameter ε , defined by

$$N = \prod_{k=1}^K \prod_{\substack{p \leq x_k \\ \text{or} \\ p < x_k}} p. \quad (11)$$

Here, if x_k is a prime p for some k , then p may or may not be a factor in the inner product. (This can occur for at most two values of k .) In other words, if $x_{k-1} < p < x_k$, then the exponent $v_p(N)$ of p in N is k , while if $p = x_k$, the exponent may be k or $k-1$. In particular, if N is the *largest* CA number of parameter ε , then

$$F(p, 1) = \varepsilon \implies P(N) = p. \quad (12)$$

Note that, since if $\varepsilon \notin \mathcal{E}$, then x_k is not prime, formula (10) gives the same value as (11). Therefore, for any ε , formula (11) gives all the possible values of a CA number N of parameter ε . (Thus N is a product of “primorials” [17, Sequence A002110].)

3 Six lemmas

The case $k = 2$ of the following lemma was proved in [14, p. 190].

Lemma 1 *For $k \geq 2$, we have the upper bound*

$$x_k < (kx)^{1/k}.$$

Proof Since the function $t \mapsto F(t, k)$ is strictly decreasing on $1 < t < \infty$, to prove that $x_k < z := (kx)^{1/k}$, it suffices to show $F(z, k) < F(x_k, k)$. As (7) and (8) imply $F(x_k, k) = \varepsilon = F(x, 1)$, this reduces to showing $F(z, k) < F(x, 1)$.

Since $z > 1$ and $k \geq 2$, we have

$$\begin{aligned} F(z, k) &= \log \left(1 + \frac{1}{z + z^2 + \dots + z^k} \right) \frac{1}{\log z} \\ &< \frac{1}{(z + z^2 + \dots + z^k) \log z} = \frac{k}{(z + z^2 + \dots + z^k) \log kx} \\ &< \frac{k}{(k - 1 + z^k) \log x} \leq \frac{k}{\left(\frac{k}{2} + kx\right) \log x} = \frac{1}{\left(x + \frac{1}{2}\right) \log x} \\ &< \log \left(1 + \frac{1}{x} \right) \frac{1}{\log x} = F(x, 1), \end{aligned}$$

using the lower bound $\log \left(1 + \frac{1}{t} \right) > \left(t + \frac{1}{2} \right)^{-1}$, valid for $t > 0$. This proves the desired inequality. \square

In the proof of Theorem 5 (iii), we will need the following result (see [11, Lemma 4]).

Lemma 2 *Given a CA number N_0 of parameter ε_0 , let $N > N_0$ be a number satisfying*

$$n \geq N_0 \implies \frac{\sigma(n)}{n^{1+\varepsilon}} \leq \frac{\sigma(N)}{N^{1+\varepsilon}} \quad (13)$$

for some fixed $\varepsilon > 0$. Then N is CA of parameter ε .

Proof Since N_0 is CA of parameter ε_0 , we have

$$\frac{\sigma(N)}{\sigma(N_0)} \leq \left(\frac{N}{N_0} \right)^{1+\varepsilon_0}.$$

On the other hand, (13) yields

$$\frac{\sigma(N)}{\sigma(N_0)} \geq \left(\frac{N}{N_0} \right)^{1+\varepsilon}.$$

Hence $\varepsilon \leq \varepsilon_0$.

In view of (13), to prove that N is CA of parameter ε , we only need to show that

$$n < N_0 \implies \frac{\sigma(n)}{n^{1+\varepsilon}} \leq \frac{\sigma(N)}{N^{1+\varepsilon}}.$$

If $n < N_0$, then since N_0 is CA and (13) holds, we have

$$\frac{\sigma(n)}{n^{1+\varepsilon}} = \frac{\sigma(n)n^{\varepsilon_0-\varepsilon}}{n^{1+\varepsilon_0}} \leq \frac{\sigma(N_0)n^{\varepsilon_0-\varepsilon}}{N_0^{1+\varepsilon_0}} \leq \frac{\sigma(N_0)N_0^{\varepsilon_0-\varepsilon}}{N_0^{1+\varepsilon_0}} = \frac{\sigma(N_0)}{N_0^{1+\varepsilon}} \leq \frac{\sigma(N)}{N^{1+\varepsilon}}.$$

This completes the proof of Lemma 2. \square

The next lemma provides an estimate for a CA number of parameter ε .

Lemma 3 *Let N be a CA number of parameter $\varepsilon < F(2, 1) = \log(3/2)/\log 2$ and define $x = x(\varepsilon)$ by (8).*

(i). *Then*

$$\log N \leq \theta(x) + c\sqrt{x}$$

for some constant $c > 0$.

(ii). *Moreover, if N is the largest CA number¹ of parameter ε , then*

$$\theta(x) \leq \log N \leq \theta(x) + c\sqrt{x}.$$

Proof (i). It follows from formula (11) for N that if x_k is defined by (7), then

$$\log N \leq \theta(x_1) + \theta(x_2) + \cdots + \theta(x_K), \quad (14)$$

where K is the largest integer such that $x_K \geq 2$. (Note that $v_2(N) = K$ or $K - 1$, and that $\varepsilon < F(2, 1)$ implies $x > 2$ and $K \geq 1$.)

As $t \mapsto F(t, k)$ is decreasing and (7) holds, we have

$$F(2, K) \geq F(x_K, K) = \varepsilon = F(x_{K+1}, K+1) > F(2, K+1).$$

On the other hand,

$$F(2, K) = \log \left(1 + \frac{1}{2^{K+1}-2} \right) \frac{1}{\log 2} < \frac{1}{(2^{K+1}-2)\log 2} \leq \frac{1}{2^K \log 2} < \frac{2}{2^K}$$

and, from (8),

$$\varepsilon = \frac{\log(1 + \frac{1}{x})}{\log x} > \frac{1}{(x+1)\log x} \geq \frac{1}{(x+1)(x-1)} > \frac{1}{x^2}.$$

Thus

$$\frac{2}{2^K} > F(2, K) \geq \varepsilon > \frac{1}{x^2},$$

implying

$$K < 1 + \frac{2}{\log 2} \log x. \quad (15)$$

Since $k \mapsto x_k$ is decreasing, from (14) we have (compare [13, equation (368)])

$$\log N \leq \theta(x_1) + \theta(x_2) + K\theta(x_3).$$

¹ Note that Ramanujan's definition of CA number of parameter ε in [13] is not exactly the same as that of Robin in [14, pp. 189–190]. Ramanujan's definition corresponds to the *largest* CA number of parameter ε for Robin.

Using $x_2 \leq \sqrt{2x}$ and $x_3 \leq \sqrt[3]{3x}$ (from Lemma 1), together with (15) and the Prime Number Theorem in the form $\theta(t) \sim t$, we deduce (i).

(ii). From (11), the largest CA number of parameter ε is

$$N = \prod_{k=1}^K \prod_{p \leq x_k} p$$

which implies $\theta(x) \leq \log N$, and (ii) follows from (i). \square

In the next lemma, we recall the oscillations of Chebychev's function θ studied by Littlewood.

Lemma 4 *There exists a constant $c > 0$ such that for infinitely many primes p we have*

$$\theta(p) < p - c\sqrt{p} \log \log \log p, \quad (16)$$

and for infinitely many other primes p we have

$$\theta(p) > p + c\sqrt{p} \log \log \log p. \quad (17)$$

Proof From Littlewood's theorem (see [10]), we know that there exists a constant $c' > 0$ such that for a sequence of values of x going to infinity we have

$$\theta(x) < x - c'\sqrt{x} \log \log \log x, \quad (18)$$

and for a sequence of values of x' going to infinity we have

$$x' + c'\sqrt{x'} \log \log \log x' < \theta(x'). \quad (19)$$

Let us suppose first that x is large enough and satisfies (18). If $x = p$ is prime, then (18) implies (16). Now assume x is not prime, and let p be the prime following x . As the function $t \mapsto t - c'\sqrt{t} \log \log \log t$ is increasing, we get

$$\begin{aligned} \theta(p) &= \theta(x) + \log p < x - c'\sqrt{x} \log \log \log x + \log p \\ &< p - c'\sqrt{p} \log \log \log p + \log p, \end{aligned}$$

which implies (16) with $c < c'$ for x large enough.

The proof of (17) is easier. Let x' satisfy (19) and choose the largest prime $p \leq x'$. For $c \leq c'$, we have

$$\theta(p) = \theta(x') > x' + c'\sqrt{x'} \log \log \log x' > p + c\sqrt{p} \log \log \log p,$$

which proves (17). \square

Lemma 5 *Chebychev's function $\theta(x)$ satisfies*

$$\theta(x) \leq (1 + \alpha)x,$$

where

$$\alpha = \alpha(x) := \begin{cases} 0 & \text{if } x \leq 8 \cdot 10^{11}, \\ \frac{1}{36260} < 0.000028 & \text{otherwise.} \end{cases}$$

Proof Schoenfeld (cf. [16, p. 360]) proved $\theta(x) \leq 1.000081x$ for all x , and he mentioned that Brent had checked that $\theta(x) < x$ for $x < 10^{11}$. The stronger results stated here are due to Dusart—see [5, p. 2 and Table 6.6]. \square

Lemma 6 *Let ε be a positive real number. For $t > e$, let us set*

$$g(t) = g_\varepsilon(t) := \varepsilon \log t - \log \log \log t. \quad (20)$$

Then there exists a unique real number $t_0 = t_0(\varepsilon) > e$ such that

$$\frac{1}{\log t_0 \log \log t_0} = \varepsilon. \quad (21)$$

Moreover, $g(t)$ is decreasing for $e < t < t_0$ and increasing for $t > t_0$.

Proof The derivative of g is

$$g'(t) = \frac{1}{t} \left(\varepsilon - \frac{1}{\log t \log \log t} \right).$$

For $t > e$, both $\log t$ and $\log \log t$ are positive and increasing. Moreover, the function $t \mapsto 1/(\log t \log \log t)$ is a decreasing bijection from $(e, +\infty)$ onto $(0, +\infty)$. Therefore, one can define $t_0 > e$ by (21).

Then we have $g'(t) < 0$ for $e < t < t_0$, and $g'(t) > 0$ for $t > t_0$, which completes the proof of Lemma 6. \square

4 GA2 numbers

We first study GA2 numbers. Compare the following result on them with Theorem 4 on extraordinary numbers.

Theorem 5 (i). *The set of GA2 numbers ≤ 5040 is*

$$\mathcal{R}_2 := \{3, 4, 5, 6, 8, 10, 12, 18, 24, 36, 48, 60, 72, 120, 180, 240, 360, 2520, 5040\}.$$

(ii). *If the Riemann Hypothesis is true, then no GA2 number exceeds 5040.*

(iii). *If the Riemann Hypothesis is false, then infinitely many GA2 numbers exist; moreover, the inequality*

$$\mu := \max\{G(n) : n > 5040\} > e^\gamma$$

holds, and any integer $A > 5040$ for which $G(A) = \mu$ is both GA2 and CA.

Proof (i). Setting

$$\mathcal{R}' := \{N \leq 5040 : N \text{ is GA2}\},$$

we have to prove that $\mathcal{R}' = \mathcal{R}_2$.

To show $\mathcal{R}' \subset \mathcal{R}_2$, choose $N \in \mathcal{R}'$. From (3), we have $G(N) \geq e^\gamma$, so that

$$\begin{aligned} N \in \mathcal{R} &:= \{r \leq 5040 : G(r) \geq e^\gamma\} \\ &= \{3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, \\ &\quad 360, 720, 840, 2520, 5040\}, \end{aligned}$$

by calculating the “ r ” column of Table 1. To show that N lies in the subset $\mathcal{R}_2 \subset \mathcal{R}$, define for $r \in \mathcal{R}$ the integer

$$a(r) := \begin{cases} \min \mathcal{A}_r, & \text{if } \mathcal{A}_r := \{a : G(ar) > G(r), ar \in \mathcal{R}\} \neq \emptyset, \\ 0, & \text{if } \mathcal{A}_r = \emptyset. \end{cases}$$

A computation (see the “ $a(r)$ ” column of Table 1) shows that

$$\{r \in \mathcal{R} : \mathcal{A}_r \neq \emptyset\} = \{9, 16, 20, 30, 84, 720, 840\}. \quad (22)$$

Since N is GA2, it must lie in the complement

$$\mathcal{R} \setminus \{9, 16, 20, 30, 84, 720, 840\} = \mathcal{R}_2. \quad (23)$$

This shows $\mathcal{R}' \subset \mathcal{R}_2$.

To prove $\mathcal{R}_2 \subset \mathcal{R}'$, choose $r \in \mathcal{R}_2$. To get $r \in \mathcal{R}'$, we need to show $G(r) \geq G(ar)$, for any multiple ar of r . We consider two cases.

Case 1: $ar \leq 5040$. If $ar \in \mathcal{R}$, then since $r \in \mathcal{R}_2$, relations (23) and (22) imply $G(ar) \leq G(r)$. On the other hand, if $ar \notin \mathcal{R}$, then we have $G(ar) < e^\gamma \leq G(r)$. Thus $G(r) \geq G(ar)$ whenever $ar \leq 5040$.

Before considering Case 2, we recall that in [14, p. 204 (c)] Robin proved that if C is the largest CA number with $P(C) < 20000$, then there is no counterexample $\leq C$ to his inequality (1). From the property (5) of CA numbers, we have $\log C \geq \theta(20000)$, where $\theta(x)$ is Chebychev’s function.

We also recall that in [16, p. 359, Corollary 2], Schoenfeld proved that

$$\theta(x) > x - \frac{x}{8 \log x} \quad (x \geq 19421).$$

A calculation then gives the inequalities

$$\theta(20000) > 20000 - \frac{20000}{8 \log 20000} > 19747$$

which, together with Robin’s result on C , yield the implication

$$5040 < n < e^{19747} \implies G(n) < e^\gamma. \quad (24)$$

Case 2: $ar > 5040$. If $\log ar < 19747$, then (24) gives $G(ar) < e^\gamma \leq G(r)$. On the other hand, if $\log ar \geq 19747$, then from (2) we get

$$G(ar) < e^\gamma + \frac{0.6483}{(\log 19747)^2} = 1.787\dots < 1.790\dots = \min_{r' \in \mathcal{R}_2} G(r') \leq G(r).$$

Thus $G(r) \geq G(ar)$ whenever $ar > 5040$.

This shows that, in both Cases 1 and 2, all elements r of \mathcal{R}_2 are GA2 numbers, so that $\mathcal{R}_2 \subset \mathcal{R}'$. Finally, since we already have $\mathcal{R}_2 \supset \mathcal{R}'$, we get $\mathcal{R}_2 = \mathcal{R}'$. This proves (i).

(ii). If RH holds, then by Robin's theorem no number $n > 5040$ exists with $G(n) \geq e^\gamma$, while from (3) a GA2 number N must satisfy $G(N) \geq e^\gamma$.

(iii). Let us assume that RH fails. Set

$$\Theta := \sup_{\zeta(\rho)=0} \Re(\rho)$$

so that

$$1/2 < \Theta \leq 1.$$

Let N denote a CA number of parameter ε , and define $x = x(\varepsilon)$ by (8). Now if $p := P(N)$ and if p^+ is the prime following p , then from (11) we have

$$p \leq x_1 = x \leq p^+,$$

which implies $x \sim p$ as $N \rightarrow \infty$. Further, from (6), we get $p \sim \log N$, which implies

$$x \sim \log N \quad (N \rightarrow \infty).$$

In [15, p. 241], it is proved that as $N \rightarrow \infty$

$$G(N) = e^\gamma \left(1 + \Omega_+ \left(x^{-b} \right) \right) \quad (1 - \Theta < b < 1/2)$$

which implies that

$$G(N) = e^\gamma \left(1 + \Omega_+ \left((\log N)^{-b} \right) \right) \quad (1 - \Theta < b < 1/2).$$

(Here the notation " $f(N) = \Omega_+(g(N))$ as $N \rightarrow \infty$ " means that $f(N) > g(N)$ infinitely often, and should not be confused with the notation $\Omega(n)$ in §1.1.) Therefore, there exist infinitely many CA numbers N satisfying $G(N) > e^\gamma$, and, for all t , we have $\max_{n \geq t} G(n) > e^\gamma$.

Now we construct sequences A_1, A_2, \dots and A'_1, A'_2, \dots , as follows. Let A_1 (resp., A'_1) be the smallest (resp., largest)² integer > 5040 such that $G(A_1) = G(A'_1) = \mu$.

Given $i \geq 2$, assume that A_1, A_2, \dots, A_{i-1} and $A'_1, A'_2, \dots, A'_{i-1}$ have been defined. Set $\mu_i := \max_{n > A'_{i-1}} G(n)$ and let A_i (resp., A'_i) be the smallest (resp., largest) integer $> A'_{i-1}$ with $G(A_i) = G(A'_i) = \mu_i$. Since we have $\mu_i > e^\gamma = \limsup G(n)$, infinitely many A_i 's can be found. The numbers A_i are such that

$$n > A_i \implies G(n) \leq G(A_i)$$

and, therefore, are GA2.

In the same way, A is proved to be GA2, using $A > 5040$ and $G(A) = \mu$. To show that A is CA, we apply Lemma 2 with $N_0 = 55040$, $\varepsilon_0 = 0.03$, $N = A$, and

² It is highly probable that $A_1 = A'_1$. A difficult question is whether G is injective.

$\varepsilon = 1/(\log A \log \log A)$; since A is GA2 and $A > 5040$, from (24) and (3) we obtain that $N = A > e^{19747} > N_0$. For $n \geq N_0$, from the definition of A we have $G(n) \leq G(A)$. Since $e < N_0 < A$ holds, it follows from Lemma 6 that, on the interval $[N_0, +\infty)$, the function $g(t)$ (defined by (20)) attains its minimum at $t = A$. Thus, for $n \geq N_0$, we have

$$\frac{\sigma(n)}{n^{1+\varepsilon}} = G(n) \frac{\log \log n}{n^\varepsilon} = G(n) e^{-g(n)} \leq G(A) e^{-g(n)} \leq G(A) e^{-g(A)} = \frac{\sigma(A)}{A^{1+\varepsilon}}$$

and so (13) holds. Applying Lemma 2 completes the proof of (iii). \square

Here is a corollary of the proof of Theorem 5.

Corollary 1 *There is no GA2 or extraordinary number between 5041 and 10^{8576} .*

Proof Since $10^{8576} < e^{19747}$, this follows from (24). \square

5 Comparison between CA and GA1 numbers

In this section, we study GA1 numbers. We begin by comparing them with CA numbers.

5.1 CA and GA1

By revisiting the proof of [15, Theorem 3, p. 242], we shall prove the following results.

Lemma 7 *Let N be a CA number of parameter $\varepsilon > 0$ and assume that $p := P(N) \geq 5$. If*

$$\varepsilon > \frac{1}{\log(N/p) \log \log(N/p)}, \quad (25)$$

then N is also a GA1 number.

Proof Let q be a prime factor of N . It follows from (5) that $6p$ divides N and that $N/q \geq N/p \geq 6 > e$, which implies $\log \log(N/q) > \log \log e = 0$. Since N is a CA number, from (4) one has

$$\frac{\sigma(N/q)}{(N/q)^{1+\varepsilon}} \leq \frac{\sigma(N)}{N^{1+\varepsilon}}, \quad (26)$$

so that

$$\frac{\sigma(N/q)}{\sigma(N)} \leq \frac{1}{q^{1+\varepsilon}}. \quad (27)$$

Since $\log \log N$ and $\log \log(N/q)$ are positive, it follows that

$$\frac{G(N/q)}{G(N)} \leq \frac{\log \log N}{q^\varepsilon \log \log(N/q)} = \frac{(N/q)^\varepsilon \log \log N}{N^\varepsilon \log \log(N/q)} = \exp(g(N/q) - g(N)), \quad (28)$$

where $g(t)$ is defined by (20). By Lemma 6, using (21) to define $t_0 > e$, we have that $g(t)$ is increasing for $t > t_0$. Now from (25) we deduce that

$$e < t_0 < \frac{N}{p} \leq \frac{N}{q} < N$$

and from (28) we get $G(N/q) < G(N)$. This shows that N is GA1. \square

Theorem 6 *Infinitely many CA numbers are GA1.*

Proof Choose a sufficiently large prime p satisfying (17), and set $\varepsilon := F(p, 1)$ (so that $x = p$, by (8)). Let N be the largest CA number of parameter ε (so that p divides N , by (12)). From Lemma 3 part (ii) and (17), we get

$$\log N \geq \theta(x) = \theta(p) > p + c\sqrt{p} \log \log \log p,$$

so that

$$\log(N/p) > p + c\sqrt{p} \log \log \log p - \log p > p + 1.$$

Using the lower bound $\log(1+t) \geq t/(1+t)$, we get

$$\begin{aligned} \varepsilon = F(p, 1) &= \frac{\log\left(1 + \frac{1}{p}\right)}{\log p} \geq \frac{1}{(p+1) \log p} \\ &> \frac{1}{(p+1) \log(p+1)} > \frac{1}{\log(N/p) \log \log(N/p)} \end{aligned}$$

and Lemma 7 implies N is GA1. Since, by Lemma 4, there are infinitely many primes p satisfying (17), the theorem is proved. \square

5.2 CA and not GA1

To study CA numbers that are not GA1, we need a lemma.

Lemma 8 *Given any prime $p \geq 3$, let N be the largest CA number of parameter $\varepsilon := F(p, 1)$. If*

$$\varepsilon < \frac{1}{\log N \log \log N}, \quad (29)$$

then N is not GA1.

Proof As $\varepsilon = F(p, 1)$, we have $p^\varepsilon = (p+1)/p = \sigma(p)/p$. Hence, by (12), inequality (26) becomes an equality when $q = p$, and so do inequalities (27) and (28). Therefore, with g and t_0 defined by (20) and (21) as in the proof of Lemma 7, we get that

$$\frac{G(N/p)}{G(N)} = \exp(g(N/p) - g(N))$$

and, from Lemma 6, that $g(t)$ is decreasing for $t < t_0$. Then the inequality (29) implies $N/p < N < t_0$, so that $G(N) < G(N/p)$. Thus N is not GA1. \square

The CA numbers N such that $P(N) \in \{2, 3, 5, 7, 11, 13, 29, 59, 149\}$ are not GA1. There are two CA numbers such that $P(N) = 23$; the larger one is not GA1, while the smaller one is GA1. All other CA numbers satisfying $P(N) < 300$ are GA1. (These statements follow by computing all CA numbers N with $P(N) < 300$, and calculating those that are GA1—see §7.)

Theorem 7 *Infinitely many CA numbers are not GA1.*

Proof Choose a sufficiently large prime p satisfying (16), and set $\varepsilon := F(p, 1)$ (so that, from (8), $x = p$). Let N be the largest CA number of parameter ε (so that, from (12), $p = P(N)$). From Lemma 3 part (i) and (16), we get

$$\log N \leq \theta(p) + c\sqrt{p} < p - c\sqrt{p} \log \log \log p + c\sqrt{p} < p,$$

and so

$$\varepsilon = \frac{\log(1 + \frac{1}{p})}{\log p} < \frac{1}{p \log p} < \frac{1}{\log N \log \log N}.$$

Then Lemma 8 implies N is not GA1. Since there are infinitely many primes p satisfying (16), the theorem is proved. \square

5.3 Odd GA1 numbers

We show that there are infinitely many odd GA1 numbers, and we compute the smallest one.

Let us denote by $\mathcal{P}_0 = \{2, 3, 5, 7, 11, 13, 17, \dots\}$ the set of all primes, and by \mathcal{P} a subset of \mathcal{P}_0 . To \mathcal{P} , we attach the set

$$\mathcal{N}_{\mathcal{P}} := \{n \geq 1 : p \mid n \implies p \in \mathcal{P}\}$$

and the function

$$\theta_{\mathcal{P}}(x) := \sum_{p \in \mathcal{P}, p \leq x} \log p.$$

A number $N \in \mathcal{N}_{\mathcal{P}}$ is said to be *colossally abundant relative to \mathcal{P}* (for short, $\text{CA}_{\mathcal{P}}$) if there exists $\varepsilon > 0$ such that

$$\frac{\sigma(N)}{N^{1+\varepsilon}} \geq \frac{\sigma(n)}{n^{1+\varepsilon}} \quad (n \in \mathcal{N}_{\mathcal{P}}).$$

For an ordinary CA number $M = \prod_{p \in \mathcal{P}_0} p^{\alpha_p}$ of parameter ε , the factor $N = \prod_{p \in \mathcal{P}} p^{\alpha_p}$ is $\text{CA}_{\mathcal{P}}$, for the same parameter ε , and all $\text{CA}_{\mathcal{P}}$ numbers can be obtained in this way.

Theorem 8 *There exist infinitely many odd GA1 numbers.*

Proof First, we observe that Lemma 7 remains valid if we replace CA with $CA_{\mathcal{P}}$, for any set \mathcal{P} with at least 2 elements.

We set $\mathcal{P} = \mathcal{P}_0 \setminus \{2\}$. The proof of Theorem 6 remains essentially valid. We just have to change the lower bound for $\log N$ to

$$\log N \geq \theta_{\mathcal{P}}(p) = \theta(p) - \log 2$$

and the inequality $\log(N/p) > p + 1$ still holds, so that we may conclude that N is GA1. \square

The smallest $CA_{\mathcal{P}_0 \setminus \{2\}}$ number that is GA1 is

$$\begin{aligned} \omega &:= 1058462574572984015114271643676625 \\ &= 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73. \end{aligned}$$

From our computation (see §7.5), ω is also the smallest odd GA1 number.

Corollary 2 *There exist infinitely many GA1 numbers that are not SA.*

Proof This follows immediately from (5) and Theorem 8. \square

Of course, the proof of Theorem 8 works for any set of primes \mathcal{P} such that $\mathcal{P}_0 \setminus \mathcal{P}$ is finite.

6 Prime factors of GA1 numbers

Here we study prime factors of proper GA1 numbers.

6.1 An upper bound

We need the following upper bound.

Theorem 9 *Given a GA1 number N with $\Omega(N) \geq 3$, let p be a prime factor of N . Then for any positive integer $r \leq v_p(N)$ we have*

$$p \leq (r \log N)^{1/r} \leq \log N.$$

Proof We have $G(N/p) \leq G(N)$, which implies

$$\frac{\sigma(N/p)N}{(N/p)\sigma(N)} \leq \frac{\log \log(N/p)}{\log \log N} = \frac{\log(\log N - \log p)}{\log \log N}. \quad (30)$$

Note that $\log \log N > \log \log(N/p) \geq \log \log 4 > 0$. We also have

$$\begin{aligned} \log(\log N - \log p) &= \log \left(\log N \left(1 - \frac{\log p}{\log N} \right) \right) \\ &= \log \log N + \log \left(1 - \frac{\log p}{\log N} \right) \end{aligned}$$

so that

$$\frac{\log(\log N - \log p)}{\log \log N} = 1 - \frac{-\log\left(1 - \frac{\log p}{\log N}\right)}{\log \log N}. \quad (31)$$

Further, setting $v = v_p(N)$, the left side of (30) can be written as

$$\begin{aligned} \frac{\sigma(N/p)N}{(N/p)\sigma(N)} &= p \frac{1 + p + \dots + p^{v-1}}{1 + p + \dots + p^v} \\ &= 1 - \frac{1}{1 + p + \dots + p^v} \geq 1 - \frac{1}{1 + p + \dots + p^r}. \end{aligned} \quad (32)$$

From (30), (31), and (32), one deduces

$$p^r \leq 1 + p + \dots + p^r \leq \frac{\log \log N}{-\log\left(1 - \frac{\log p}{\log N}\right)} \leq \frac{\log N \log \log N}{\log p} \quad (33)$$

which yields

$$p^r \log p \leq \log N \log \log N. \quad (34)$$

Let us assume, *ab absurdum*, that $p > (r \log N)^{1/r}$. Then we would have

$$p^r \log p > (r \log N)^{\frac{1}{r}} \log(r \log N) = \log N \log(r \log N) \geq \log N \log \log N$$

contradicting (34). Therefore, $p \leq (r \log N)^{1/r}$ holds. Finally, by calculus, $(r \log N)^{1/r}$ is decreasing for $r \geq 1$ (because $\Omega(N) \geq 3$ implies $N \geq 8$ and $\log N > 2$) and the theorem follows. \square

6.2 Study of $\Omega(N)$ where N is GA1

We show that there are only finitely many proper GA1 numbers N that have a fixed value of $\Omega(N)$.

Theorem 10 *If $k \geq 3$, then*

$$\Pi_k := \#\{N : N \text{ is GA1 and } \Omega(N) = k\} < \infty.$$

Proof For a GA1 number N with $\Omega(N) = k > 2$, let us write $N = p_1 p_2 \dots p_k$ with $p_1 \leq p_2 \leq \dots \leq p_k$. We have $N \leq p_k^k$, so that $p_k \geq N^{1/k}$ holds. But Theorem 9 yields $p_k \leq \log N$, whence

$$\frac{\log N}{\log \log N} \leq k$$

and N is bounded. Thus Π_k is finite. \square

Since $\frac{\log 10^{60}}{\log \log 10^{60}} = 28.03\dots$, a table of GA1 numbers up to 10^{60} (see §7) allows us to calculate Π_k for $k \leq 28$.

We have $\Pi_k = 0$ if $3 \leq k \leq 12$, and the following table gives Π_k when $13 \leq k \leq 28$.

$k =$	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\Pi_k =$	2	4	2	1	1	2	4	1	2	3	7	7	7	1	4	7

6.3 The exponent of the largest prime factor

First, we observe that the function $t \mapsto 2^t/t$ is an increasing bijection of the interval $[2, +\infty)$ to itself. Let us introduce the inverse function h defined for $x \geq 2$ by

$$h(x) = t \iff x = \frac{2^t}{t}. \quad (35)$$

We shall need the following lemma.

Lemma 9 *Let x satisfy $x \geq 2$. Then we have $2 \leq h(x) \leq 3.08 \log x$.*

Proof The lower bound results from the definition of h . Let us set $t = h(x)$, so that $x = 2^t/t$. By noting that $(\log t)/t \leq 1/e$ holds, we get

$$\frac{h(x)}{\log x} = \frac{t}{t \log 2 - \log t} = \frac{1}{\log 2 - (\log t)/t} \leq \frac{1}{\log 2 - 1/e} = 3.0743 \dots$$

which proves Lemma 9. \square

Theorem 11 *Let N be a GA1 number with $\Omega(N) \geq 3$. Set $R = h(\log N)$, so that $2^R/R = \log N$. Then N divides the number $M = M(N)$ defined by*

$$M := \prod_{r=1}^{\lfloor R \rfloor} \prod_{((r+1) \log N)^{1/(r+1)} < p \leq (r \log N)^{1/r}} p^r = \prod_{r=1}^{\lfloor R \rfloor} \prod_{p \leq (r \log N)^{1/r}} p. \quad (36)$$

Proof As $(r \log N)^{1/r}$ is a decreasing function of r , this follows from Theorem 9. \square

For example, if $N = v = 183783600$, we compute $R = h(\log v) = 7.072 \dots$ and find that $M = 72v$.

Theorem 11 allows the computation of proper GA1 numbers—see §7.2 and §7.5.

For the exponent $v_p(N)$ of a prime p in the usual factorization of N , Theorem 11 provides the upper bound $v_p(N) \leq v_p(M)$, which only depends on the size of N .

We now study the exponent of the largest prime factor of a GA1 number.

Theorem 12 *Let N be a GA1 number with $\Omega(N) \geq 3$, and let $p = P(N)$ be its largest prime factor. Then $v_p(N) = 1$.*

Proof Suppose on the contrary that $v := v_p(N) \geq 2$. Then Theorem 9 implies that N divides the number

$$M_v = M_v(N) := \left(\prod_{p \leq (v \log N)^{1/v}} p \right)^v \prod_{r=v+1}^{\lfloor R \rfloor} \prod_{p \leq (r \log N)^{1/r}} p$$

with R defined by $2^R/R = \log N$. Thus, from the function $r \mapsto (r \log N)^{1/r}$ being decreasing,

$$\begin{aligned} \log N &\leq \log M_v = v \theta(v \log N)^{1/v} + \sum_{r=v+1}^{\lfloor R \rfloor} \theta((r \log N)^{1/r}) \\ &\leq 2\theta((2 \log N)^{1/2}) + R\theta((3 \log N)^{1/3}). \end{aligned}$$

From Lemmas 9 and 5, it follows that

$$\begin{aligned}\log N &\leq 1.000028 \left(2\sqrt{2\log N} + 3.08 \log \log N (3\log N)^{1/3} \right) \\ &\leq 2.83\sqrt{\log N} + 4.45 \log \log N (\log N)^{1/3}.\end{aligned}$$

Therefore, we have

$$\frac{2.83}{\sqrt{\log N}} + \frac{4.45}{(\log N)^{2/3} \log \log N} \geq 1$$

which implies $\log N \leq 15.03$, $N \leq N_0 := 3336369$ and $R \leq 6.65$, so that N must divide $M_v(N_0)$ for some v in the range $2 \leq v \leq 6$. But the table

$v =$	2	3	4	5	6
$(v \log N_0)^{1/v} =$	5.48	3.56	2.78	2.37	2.12
$M_v(N_0) =$	$43200 = 2^6 3^3 5^2$	$1728 = 2^6 3^3$	$64 = 2^6$	64	64

shows that if $v \geq 2$, then the number $M_v(N_0)$ divides $M_2(N_0) = 43200$, contradicting the easily-checked fact that none of the 84 divisors of 43200 is a proper GA1 number. (In fact, we will see in §7.5 that there is no proper GA1 number < 183783600 .) This proves the theorem. \square

6.4 The largest prime factor of a GA1 number

For a GA1 number, we now study the largest prime factor itself.

Theorem 13 *For GA1 numbers N with $\Omega(N) \geq 3$, the largest prime factor satisfies*

$$P(N) \sim \log N \quad (N \rightarrow \infty).$$

Proof Let N be a GA1 number satisfying $\Omega(N) \geq 3$ and let $p := P(N)$ be its largest prime factor. From Theorem 9, we know that

$$p \leq \log N. \quad (37)$$

It remains to get a lower bound for p . The proof resembles that of Theorem 12.

Since N divides M given by (36) and $p = P(N)$, by Lemma 5 we have

$$\begin{aligned}\log N &\leq \theta(p) + \sum_{r=2}^R \theta((r \log N)^{1/r}) \\ &\leq \theta(p) + \theta((2 \log N)^{1/2}) + \mathcal{O}(R(\log N)^{1/3}) \\ &\leq \theta(p) + \mathcal{O}(\sqrt{\log N}).\end{aligned} \quad (38)$$

From the Prime Number Theorem and from (37), we get

$$\theta(p) = p + \mathcal{O}(p \exp(-c\sqrt{\log p})) = p + \mathcal{O}(\log N \exp(-c\sqrt{\log \log N})).$$

Therefore, (38) becomes

$$\log N \leq p + \mathcal{O}(\log N \exp(-c\sqrt{\log \log N})),$$

which, together with (37), completes the proof of the theorem. \square

7 Computation of GA1 numbers

In this section we give several versions of an algorithm to compute GA1 numbers.

7.1 The Gronwall quotient

We begin with a lemma and a definition.

Lemma 10 *Let n be a positive integer with $\Omega(n) \geq 3$. Let q and p be prime factors of n satisfying $q < p$ and $v_q(n) \leq v_p(n)$. Then we have*

$$G(n/q) < G(n/p).$$

Proof We have

$$\begin{aligned} \frac{\sigma(n/q)/(n/q)}{\sigma(n)/n} &= q \frac{\sigma(n/q)}{\sigma(n)} = \frac{q + \dots + q^{v_q(n)}}{1 + q + \dots + q^{v_q(n)}} = 1 - \frac{1}{1 + q + \dots + q^{v_q(n)}} \\ &\leq 1 - \frac{1}{1 + p + \dots + p^{v_p(n)}} = \frac{\sigma(n/p)/(n/p)}{\sigma(n)/n} \end{aligned}$$

which implies $\frac{\sigma(n/q)}{n/q} \leq \frac{\sigma(n/p)}{n/p}$.

The lemma follows from $\log \log(n/q) > \log \log(n/p)$, since $\Omega(n) \geq 3$ implies $\log \log(n/p) \geq \log \log 4 > 0$. \square

We define the *Gronwall quotient* $Q(n)$ of a composite integer n to be the number

$$Q(n) := \max_{\substack{p|n \\ p \text{ prime}}} \frac{G(n/p)}{G(n)} = \max_{\substack{p|n \\ p \text{ prime}}} \frac{p^{v_p(n)+1} - p}{p^{v_p(n)+1} - 1} \frac{\log \log n}{\log \log(n/p)}.$$

GA1 numbers N are characterized by $Q(N) \leq 1$. For example, the “ $Q(r)$ ” column in Table 1 shows that the only GA1 number $r \in \mathcal{R}$ is $r = 4$.

Let us introduce a subset $\mathcal{S}(n)$ of the set of the prime divisors of n . The elements of $\mathcal{S}(n)$ are defined by induction. The largest prime factor of n is the first element q_1 of $\mathcal{S}(n)$. Now let us assume that $i \geq 2$ and that the elements $q_1, q_2, \dots, q_{i-1} \in \mathcal{S}(n)$ are known.

If $v_p(n) \leq v_{q_{i-1}}(n)$ for all primes p that divide n and are smaller than q_{i-1} , then there are no further elements of $\mathcal{S}(n)$, and we have $\mathcal{S}(n) = \{q_1, q_2, \dots, q_{i-1}\}$.

If not, then the element $q_i \in \mathcal{S}(n)$ is defined as the largest prime factor of n that satisfies $q_i < q_{i-1}$ and $v_{q_i}(n) > v_{q_{i-1}}(n)$.

From Lemma 10, if $\Omega(n) \geq 3$ we get

$$Q(n) = \max_{p \in \mathcal{S}(n)} \frac{G(n/p)}{G(n)} = \max_{p \in \mathcal{S}(n)} \frac{p^{v_p(n)+1} - p}{p^{v_p(n)+1} - 1} \frac{\log \log n}{\log \log(n/p)}.$$

7.2 A first algorithm

To compute all proper GA1 numbers $N \leq x$ for a given x , we first calculate $M = M(x)$, defined by

$$M := \prod_{r=1}^{\lfloor R \rfloor} \prod_{p \leq (r \log x)^{1/r}} p,$$

with R such that $(R \log x)^{1/R} = 2$. Any GA1 number $N \leq x$ with $\Omega(N) \geq 3$ is a divisor of M (see Theorem 11).

Thus a first version of the algorithm computes all composite divisors N of M , and for each of them calculates $G(N/p)/G(N)$ for all $p \in \mathcal{S}(N)$. If for some $p \in \mathcal{S}(N)$ we have $G(N/p)/G(N) > 1$, we stop: N is not GA1. If not, we compute the Gronwall quotient $Q(N)$ (which involves *all* primes p dividing N): N is GA1 if and only if $Q(N) \leq 1$.³

7.3 A second algorithm

A more elaborate version of the algorithm tests only a small number of the divisors of M . First, we define

$$M_1 := \prod_{r=2}^{\lfloor R \rfloor} \prod_{p \leq (r \log x)^{1/r}} p$$

so that $M_2 := M/M_1$ is squarefree. Let us write $M_2 = p_1 p_2 \cdots p_s$ where p_1, p_2, \dots, p_s are consecutive primes in ascending order.

As a first step, we compute the set \mathcal{D}_0 of all the composite divisors of M_1 and test each of them for GA1 by the method described above.

A divisor of M whose largest prime factor is p_i is equal to $d p_i$, where d is a divisor of M whose largest prime factor is $< p_i$. Therefore, we construct by induction on $i = 1, 2, \dots, s$ the set \mathcal{D}'_i containing those divisors of M whose largest prime factor is p_i , and the set \mathcal{D}_i containing the divisors of M whose largest prime factor is $\leq p_i$. Then \mathcal{D}'_i is equal to $p_i \mathcal{D}_{i-1}$ and $\mathcal{D}_i = \mathcal{D}'_i \cup \mathcal{D}_{i-1}$. From Theorem 9, for $i = 1, 2, \dots, s$, we only have to test the elements of \mathcal{D}'_i that are greater than $\exp(p_i)$.

7.4 A third algorithm

Let us say that a divisor $d \in \mathcal{D}_i$ (with $0 \leq i < s$) is *bad* if, for every j satisfying $i < j \leq s$, all multiples of d belonging to \mathcal{D}_j are smaller than $\exp(p_j)$.

The largest multiple of d belonging to \mathcal{D}_j is $d p_{i+1} p_{i+2} \cdots p_j$, so that d is bad if and only if

$$\log d < \delta_i := \theta(p_i) + \min_{i < j \leq s} (p_j - \theta(p_j)).$$

³ To avoid roundoff errors, we carry out our computation in floating point arithmetic with 20 decimal digits and choose a small ε (typically, $\varepsilon = 10^{-5}$). In the first step, we keep the N 's satisfying $Q(N) \leq 1 + \varepsilon$. For these N 's, we start the computation again with 40 digits.

Therefore, we write $\mathcal{G}_i \subset \mathcal{D}_i$ for the set obtained from \mathcal{D}_i by removing the bad divisors, i.e., those divisors d satisfying $d < \Delta_i := \exp(\delta_i)$.

Further, we construct \mathcal{G}'_{i+1} and \mathcal{G}_{i+1} by removing from $p_{i+1}\mathcal{G}_i$ and $p_{i+1}\mathcal{G}_i \cup \mathcal{G}_i$, respectively, those divisors d that satisfy $d < \Delta_{i+1} = \exp(\delta_{i+1})$.

For $i = 1, 2, \dots, s$, it remains to test the elements of \mathcal{G}_i whose largest prime factor is equal to p_i , that is, the elements of \mathcal{G}'_i .

7.5 Results

The smallest proper GA1 number is

$$v = 183783600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17.$$

We compute that $M = M(v) = 8 \cdot 19 \cdot v$ and we find that there is no proper GA1 number $N < v$.

Using the third algorithm, we have computed all GA1 numbers $N \leq 10^{60}$ with $\Omega(N) \geq 3$.

These results as well as the Maple code can be found on the web site <http://math.univ-lyon1.fr/~nicolas/GAnumbers.html>.

We hope to present soon a fourth algorithm, more sophisticated, and able to compute GA1 numbers up to 10^{120} .

8 The number of GA1 numbers up to x

Let $Q_1(x)$ be the number of proper GA1 numbers $N \leq x$. From (36) we know that $Q_1(x)$ does not exceed the number $\tau(M)$ of divisors of

$$M = M(x) := \prod_{r=1}^{\lfloor R \rfloor} \prod_{p \leq (r \log x)^{1/r}} p$$

with $(R \log x)^{1/R} = 2$. It is easy to see that $\log M \sim \log x$ as $x \rightarrow \infty$, and from the estimation of the large values of the function τ (cf. [8] or [11]), it follows that

$$Q_1(x) \leq \exp \left(c \frac{\log x}{\log \log x} \right)$$

for some positive c . By estimating the number of *good* divisors of M (that is, divisors that are not bad—see §7.4), it might be possible to improve the above estimate.

It seems more difficult to get a lower bound for $Q_1(x)$. We hope to return to these questions in another article.

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